Stochastic Processes and their Applications 39 (1991) 107-115
North-Holland

The symmetric simple exclusion process, II: Applications

P.A. Ferrari

Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil

E. Presutti

Dipartimento Matematico, Università di Roma, "Tor Vergata", Rome, Italy

E. Scacciatelli

Dipartimento Matematico, Università di Roma, "La Sapienza", Rome, Italy

M.E. Vares

IMPA, Rio de Janeiro, Brazil

Received 4 August 1989
Revised 18 October 1990

We consider the one dimensional nearest neighbour symmetric simple exclusion process. We use the probability estimates obtained in a companion paper, Ferrari et al. (1991), to study some 'collective' properties of the particle system. In particular we give another proof of a pointwise ergodic theorem.

collective phenomena * interacting particle systems * ergodicity

1. Introduction

Stochastic processes with infinitely many interacting particles have been successfully used to model systems in such a variety of fields as in physics, chemistry, population genetics, economy, ..., as well as to obtain discrete approximations for PDE's. Numerical analysis and computer simulations are often the main tools for such investigations, since a mathematically rigorous study is in many cases hopeless. In computer simulations one looks at sample paths of the particles and measures time averages of observables (functions) of interest to deduce the equilibrium properties of the system (equilibrium averages of the functions). This procedure, which may present serious difficulties, is based on the Birkhoff ergodic theorem. Important improvements are the so-called pointwise ergodic theorems, giving sufficient conditions on an initial configuration for the convergence of the time averages for almost all realizations of the process *starting from that particular configuration.* In [5] this was studied in the context of the contact process; [2] and [1] treat the symmetric simple exclusion process.

Research partially supported by CNR MMAIT, CNPQ and IBM of Brazil.

0304-4149/91/$03.50

We shall consider only the one dimensional symmetric simple exclusion process with nearest neighbour jumps (SEP), but more general symmetric simple exclusion processes can be treated similarly. Given $\eta \in X \stackrel{\text{def}}{=} \{0,1\}^{\mathbb{Z}}$ we let P_η denote the law of the SEP starting at η at time zero, and E_η denotes the corresponding expectation. As it is well know P_η can always be constructed in the Skorohod space $\tilde{\Omega} = D([0,+\infty), X)$. We refer to [6] for basic facts about the SEP, and we shall use the same notation as in [4]. In particular $\eta(x,t)$ denotes the occupation number (0 or 1) at site x, at time t. We also denote by ν_p the product measure on X with $\nu_p\{\eta\colon \eta(x)=1\}=p$ for all $x \in \mathbb{Z}$. Finally if μ is a probability measure on X we let $P_\mu = \int \mu(\mathrm{d}\eta) P_\eta$ be the law of the SEP when $\eta(\cdot, 0)$ is distributed according to μ.

Let us start by recalling the following:

Theorem 1.1 [2, 1]. *Let $\eta \in X$ be such that*

$$\lim_{t\to+\infty} E_\eta(\eta(0,t)) = p \tag{1.1}$$

for some $p \in [0,1]$. Then a.s. P_η we have

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(\eta(s))\,\mathrm{d}s = \int f\,\mathrm{d}\nu_p \tag{1.2}$$

for all $f: X \to \mathbb{R}$ continuous. □

Remark. It is easily seen that for our particular case condition (1.1) implies $\lim_{t\to+\infty} E_\eta(\eta(x,t)) = p$ for all x, and so it is equivalent to the condition in [2] and [1]. (cf. [6, Chapter VIII]).

In several cases the 'experiments' seem to indicate a 'very fast convergence' of the time averages appearing in equation (1.2). But, after a longer time, one finds out that the average 'slowly' changes departing from its previous value, the true asymptotic value will only be reached on a much longer time scale. Such phenomenon is due to the presence of so called 'long time tails' and it occurs frequently in the analysis of systems with very many 'components'. Clearly statistical fluctuations are not responsible for such effects which, in fact, should be depressed by taking longer time averages. The phenomenon is on the contrary caused by the presence of long range space-time correlations which establish throughout the system, the so-called 'hydrodynamical modes', i.e. some distinguished functions which relax on a much longer time scale than all the other observables. For instance consider spatially extended systems of classical point particles pairwise interacting via conservative forces. The hydrodynamical modes in such a case are the 'extensive conserved quantities' namely the energy, the number of particles and the mean velocity. The value of such observables in macroscopic (suitably large) subregions of the whole volume is 'essentially constant' in some finite but not too long time interval. Physical arguments lead to conjecture that such time intervals are also so long for each subsystem to reach its own equilibrium, determined by the 'local' values of the energy, the number of particles and the mean velocity in any of the above subregions. On a longer time scale also such local equilibrium values will change because of some diffusive and convective phenomena, and in agreement with the hydrodynamic equations for the system. The existence of the 'long time tails' is related to this second time scale. Something analogous happens also in our simple model. We have in fact the following:

Theorem 1.2. *Let $\alpha \in (0, 1)$ and $\eta \in X$. Then, almost surely with respect to P_η we have that for all continuous f, and all $n > 1$,*

$$\lim_{t\to+\infty} \sup_{|x|\leq t^n} \left| t^{-\alpha} \int_t^{t+t^\alpha} \mathrm{d}s\, f(S_x\eta(s)) - \int f \,\mathrm{d}\nu_{p(x,t)} \right| = 0, \tag{1.3}$$

where for any $\xi \in X$, $S_x\xi$ denotes the configuration ξ shifted to the left by x (i.e. $S_x\xi(y) = \xi(y+x)$ for all y), and $p(x, t) = E_\eta(\eta(x, t))$.

Namely if t in eq. (1.3) is large then the time average is essentially given by an equilibrium average, which depends on 'when' (i.e. on t) and 'where' (i.e. on x) the observable is located. In fact, $\{\nu_p, p \in [0, 1]\}$ is the collection of all the extremal invariant (equilibrium) measures for the symmetric simple exclusion process, (cf. [6] for instance). To complete the hydrodynamical picture of the symmetric simple exclusion process one needs to find the equations solved by $p(x, t)$. It is easy to see, cf. [3], that there is a C^∞ function $q(r, t)$ such that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{Z}} |q(x, t) - p(x, t)| = 0, \tag{1.4}$$

and which solves the equation

$$\frac{\partial}{\partial t} q(r, t) = \frac{1}{2} \frac{\partial^2}{\partial r^2} q(r, t). \tag{1.5}$$

In this sense one can conclude that the hydrodynamical behavior of the symmetric simple exclusion process is described by a linear diffusive equation. To evidentiate the collective phenomena described so far one usually improves the statistics of his experiments by taking also space averages and by choosing a random configuration according to some suitable distribution, which simulates the initial macroscopic profile. A theoretical frame for such considerations can again be established (at a rigorous level) in the case of the symmetric simple exclusion process. For this, we assume a family of initial measures μ^ε, $\varepsilon \in (0, 1)$. For each ε, μ^ε is a product probability measure on X and

$$\mu^\varepsilon\{\eta\colon \eta(x) = 1\} = q(\varepsilon x) \tag{1.6}$$

for all $x\in\mathbb{Z}$, where q is a $\mathbb{C}^1$ function from $\mathbb{R}$ to $[0,1]$ (the initial macroscopic profile) with bounded derivative. We denote by $\mathcal{I}(\varepsilon,\beta;y)$, $y\in\mathbb{Z}$, the space interval

$$\mathcal{I}(\varepsilon,\beta;y)=\{x\in\mathbb{Z}: |x-y|\leq\tfrac{1}{2}\varepsilon^{-\beta}\}.$$

We then have the following theorem:

Theorem 1.3. *Let μ^ε be as above and given $\beta>0$ let*

$$0<\gamma<\tfrac{1}{2}\beta. \tag{1.7}$$

Fix any $T>0$, $n>1$, $k>1$ and $x_1,\dots,x_k$ pairwise distinct. Then

$$\lim_{\varepsilon\to0} P_{\mu^\varepsilon}\left[\sup_{t\leq\varepsilon^{-2}T}\ \sup_{|y|\leq\varepsilon^{-n}} \varepsilon^{\beta}\left|\sum_{x\in\mathcal{I}(\varepsilon,\beta;y)}\prod_{i=1}^{k}\{\eta(x_i+x,t)-E_{\mu^\varepsilon}[\eta(x_i+x,t)]\}\right|\geq\varepsilon^{\gamma}\right]=0. \tag{1.8}$$

To complete the picture we again recall the well known fact that for any r and τ,

$$\lim_{\varepsilon\to0} E_{\mu^\varepsilon}[\eta([\varepsilon^{-1}r],\varepsilon^{-2}\tau)]=q(r,\tau), \tag{1.9}$$

where $[a]$ denotes the integer part of a and $q(r,\tau)$ solves equation (1.5) with initial value $q(r)$. (See [3] for example.)

Remark. We can relax these assumptions by assuming some mixing condition on μ^ε and by taking more general density profiles $q(r)$, but we shall not discuss here such extensions.

2. Proofs

Proof of Theorem 1.2. We fix $\eta\in X$. It is enough to prove that for any $0<\alpha<1$, for any $n\geq1$, for any $k>1$, and any set $x_1,\dots,x_k$ of mutually distinct sites

$$\lim_{t\to+\infty,t\in\mathbb{Z}}\ \sup_{|x|\leq t^n}\left|t^{-\alpha}\int_t^{t+t^\alpha} ds\prod_{i=1}^{k}\eta(x+x_i,s)-[E_\eta(\eta(x,t))]^k\right|=0 \tag{2.1}$$

a.s. P_η.

In fact since α is positive and t^α diverges, the restriction that t is in Z becomes unimportant. By taking countable intersection of the fullsets over all n, over all k and all $x_1,\dots,x_k$ one then obtains that of Theorem 1.2. (We used that the set of linear combinations of functions $\prod \eta(x_i)$ is dense in $C(X)$, the Banach space of all continuous functions equipped with sup norm.)

We shall prove the next:

Statement. *Given any $\alpha \in (0,1)$ there is $\gamma > 0$ so that the following holds. For any $k \geqslant 1$ and for any N there is C_N so that given any $\eta \in X$ and k mutually distinct sites $x_1, \ldots, x_k$,*

$$E_\eta\left[\left\{t^{-\alpha}\int_t^{t+t^\alpha} \mathrm{d}s\left[\prod_{i=1}^k \eta(x_i, s) - \prod_{i=1}^k E_\eta(\eta(x_i, t))\right]\right\}^{2N}\right] \leqslant C_N t^{-\gamma N}. \tag{2.2}$$

From simple comparison between $E_\eta(\eta(x+x_i, t))$ and $E_\eta(\eta(x, t))$ using equation (2.11) below, and the Chebichev inequality one readily sees that the statement implies (2.1), and hence it proves Theorem 1.2. We shall prove (2.2) with

$$\gamma = \min\{\tfrac{1}{8}, \tfrac{1}{4}\alpha, \tfrac{1}{2}(1-\alpha)\}. \tag{2.3}$$

We use Fubini's theorem to rewrite the left hand side of (2.2) as

$$(2N)!\,t^{-\alpha 2N}\int_t^{t+t^\alpha} \mathrm{d}s_1 \cdots \int_{s_{2N-1}}^{t+t^\alpha} \mathrm{d}s_{2N}\, E_\eta\left[\prod_{j=1}^{2N}\left\{\prod_{i=1}^k \eta(x_i, s_j) - \prod_{i=1}^k E_\eta(\eta(x_i, t))\right\}\right]. \tag{2.4}$$

We fix $t \leqslant s_1 \leqslant \cdots \leqslant s_{2N} \leqslant t + t^\alpha$ and then we use duality, cf. [4] and references quoted in, to compute the E_η expectation. The dual process is the following: there are k particles which start at time 0 (in the dual process we reverse the time so that time 0 corresponds to s_{2N} in the original process). The k particles are at the sites $x_1, \ldots, x_k$ and move like stirring particles for a time equal to $s_{2N} - s_{2N-1}$. Let $y_1, \ldots, y_k$ be their position at such time. We then consider particles starting from $x_1, \ldots, x_k, y_1, \ldots, y_k$ (if $y_i = x_j$ for some i and j we only consider one particle starting from x_j). We let all these particles move, like stirring particles, for a time $s_{2N-1} - s_{2N-2}$. We iterate such a procedure till we reach a total time equal to $s_{2N} - s_1$.

Let $\boldsymbol{x}^{(j)}$, $j = 1, \ldots, 2N$, be the position at this last time of the particles starting from $x_1, \ldots, x_k$ at the 'dual' time $s_{2N} - s_j$. We then need to evaluate

$$t^{-\alpha 2N} E\left[E_\eta\left[\prod_{j=1}^{2N}\left\{\prod_{i=1}^k \eta(x_i^{(j)}, s_1) - \prod_{i=1}^k E_\eta[\eta(x_i, t)]\right\}\right]\right], \tag{2.5}$$

where the average E refers to the dual process of the variables x's described above. We fix $\tau \equiv t^{\alpha/2}$ and we consider the times s_i, $i = 1, \ldots, 2N$, such that $|s_i - s_{i\pm 1}| > \tau$. We shall call the labels of these times *free*. The label i is *superfree* if it is free and $\boldsymbol{x}^{(i)} \cap \boldsymbol{x}^{(j)} = \emptyset\ \forall j \neq i$.

Notice that being superfree is random since it depends on the dual process. In the sequel we shall see that the E_η expectation in (2.5) gives a contribution which goes like $t^{-\gamma M}$, if M is the number of superfree labels. The proof of (2.2) will then be completed by proving that (1) the probability (in the dual process) that there are L free labels which are not superfree goes like $t^{-\gamma L}$ and that (2) the contribution of the time integrals when there are K non free labels goes like $t^{-\gamma K}$.

Lemma 2.1. *For any N there is C_N so that the following holds. Setting*

$$\eta(\mathbf{x}^{(j)}) = \prod_{i=1}^{k} \eta(x_i^{(j)}), \tag{2.6}$$

and denoting by M the total number of superfree labels, we have

$$\left| E_\eta\left[\prod_{j=1}^{2N}\left\{\eta(\mathbf{x}^{(j)}, s_1) - \prod_{i=1}^{k} E_\eta[\eta(x_i^{(j)}, s_1)]\right\}\right]\right| \le c_N t^{-M/8}. \tag{2.7}$$

Proof. The proof is a simple consequence of Theorem 2.1 of [4]: first notice that if j is superfree then $x_i^{(j)} \ne x_{i'}^{(j')}$ if $(j, i) \ne (j', i')$ so that $\eta(x_i^{(j)})$ appears only once in each product. Take then a j which is superfree and write

$$\eta(\mathbf{x}^{(j)}, s_1) = \prod_{i=1}^{k} (\eta(x_i^{(j)}, s_1) - E_\eta[\eta(x_i^{(j)}, s_1)] + E_\eta[\eta(x_i^{(j)}, s_1)]) \tag{2.8}$$

and then expand the products and substract the term $\prod_{i=1}^{k} E_\eta(\eta(x_i^{(j)}, s_1))$. We obtain a sum of terms. In each of them there is at least one factor of the type

$$\eta(x_i^{(j)}, s_1) - E_\eta[\eta(x_i^{(j)}, s_1)] \tag{2.9}$$

for some $i \in \{1, \ldots, k\}$. By repeating this procedure for all the superfree labels we get a sum of terms where in each of them there is a product of at least M factors of the type $\eta(x_i^{(j)}, s_1) - E_\eta[\eta(x_i^{(j)}, s_1)]$. By Theorem 2.1 of [4], recalling that $s_1 > t$, the lemma follows. □

In (2.5) we have $\prod_{i=1}^{k} E_\eta[\eta(x_i, t)]$ instead of $\prod_{i=1}^{k} E_\eta(x_i^{(j)}, s_1)]$. In order to use (2.7) we need first to estimate their difference:

Lemma 2.2. *There is a constant c_1 such that if $s \in [t, t+t^\alpha]$, then*

$$|E_\eta[\eta(x_i^{(j)}, s)] - E_\eta[\eta(x_i, t)]| \le c_1[|x_i^{(j)} - x_i| + t^{\alpha/2}]t^{-1/2} \tag{2.10}$$

for all $j = 1, \ldots, 2N$ and $i = 1, \ldots, k$.

Proof. This is a consequence of simple estimates on single random walks: in fact, using duality,

$$E_\eta[\eta(x, s)] = \sum_{y \in \mathbb{Z}} \pi_s(x \to y)\eta(y), \tag{2.11}$$

where $\pi_s(x \to y)$ is the probability for a unit rate symmetric n.n random walk which starts from x to be at y at time s. From this the lemma easily follows. □

The term $x_i^{(j)} - x_i$ is estimated as follows:

Lemma 2.3. *Let P be the law of the dual process. Then, with the above notation, for any $\gamma' > \frac{1}{2}\alpha$ there are positive constants c_2 and b_1 so that*

$$\begin{aligned} &P[\exists j \in \{1, \dots, 2N\}, \exists i \in \{1, \dots, k\}: |x_i^{(j)} - x_i| > t^{\gamma'}] \\ &\quad \leq c_2 \exp\{-b_1 t^{\gamma' - \alpha/2}\}. \end{aligned} \tag{2.12}$$

Proof. We know from [3] and [4] that if $x(t)$ and $x^\circ(t)$ denote respectively the position of n stirring particles and n independent random walks and assuming that $x(0) = x^\circ(0)$, then for any $\beta > \frac{1}{4}$ there are constants c_3 and b_1' so that

$$P\left(\bigcup_{t' \geq t} [|x(t') - x^\circ(t')| > t'^{\beta}]\right) \leq c_3 \exp\{-b_1' t^{\beta - 1/4}\}.$$

The Lemma is then reduced to an estimate for independent random walks and is easily proven. □

We choose $\gamma' > \frac{1}{2}\alpha$ and such that $\frac{1}{2} - \gamma' > \gamma$, with γ as in (2.3). Then in the complement of a set of probability $c_2 \exp[-b_1 t^{\gamma' - \alpha/2}]$ the bound in (2.10) goes like $t^{-\gamma}$. By the previous lemmas we then have a contribution coming from M superfree labels which goes like $t^{-\gamma M}$. Next we estimate the probability of having k free labels which are not superfree.

Lemma 2.4. *There is c_4 so that the following holds. Let $L \geq 1$ and $j_1 < \dots < j_L$ be free labels and denote by $P(j_1 < \dots < j_L)$ the probability that $j_1 < \dots < j_L$ are not superfree. Then*

$$P(j_1 < \dots < j_L) \leq c_4 \tau^{-L/2}. \tag{2.13}$$

Proof. The basic point is the following. Let A and B be finite sets in Z. Let B_s be the random configuration at time s of the stirring particles which at time 0 were in B. Then

$$P[|B_s \cap A| = m] \leq \left\{\sum_{x \in B} P[x_s \in A]\right\}^m \leq c_5 \{s^{-1/2} |B| |A|\}^m \tag{2.14}$$

where x_s is the position at time s of a random walk which starts at time 0 from x and c_5 is a suitable constant. (2.14) is a straight consequence of lemma 4.12 of [6]. From (2.14) the estimate in (2.13) easily follows. □

So far we know that if $\gamma < \min\{\frac{1}{2}(1-\alpha), \frac{1}{8}\}$ then

$$E\left[\left|E_\eta\left[\prod_{j=1}^{2N}\left\{\prod_{i=1}^{k}\eta(x_i^{(j)}, s_1) - \prod_{i=1}^{k} E_\eta[(x_i, t)]\right\}\right]\right|\right]$$

$$\leqslant \sum_{M\leqslant L^*} c_6 t^{-\gamma M} P[\exists L^* - M \text{ free and not superfree labels}]$$

$$\leqslant \sum_{M\leqslant L^*} c_7 t^{-\gamma M-(L^*-M)\alpha/4}$$

$$\leqslant c_8 t^{-\gamma L^*}, \tag{2.15}$$

provided that $\gamma < \frac{1}{4}\alpha$. L^* denote the number of free labels and c_6, c_7 and c_8 are suitable constants.

Therefore from (2.2) and (2.4) we get that the left hand side of (2.4) is bounded by

$$c_8 t^{-\gamma L^*} + t^{-\alpha 2N}\int_t^{t+t^\alpha} \mathrm{d}s_1 \cdots \int_{s_{2N-1}}^{t+t^\alpha} \mathrm{d}s_{2N}\, 1(\{\exists 2N - L^* \text{ not free labels}\})$$

$$\leqslant c_9 t^{-\gamma L^*}(\tau/t^\alpha)^{2N-L^*}$$

for a suitable constant c_9. From this (2.2) follows, hence Theorem 1.2 is proven. □

Proof of Theorem 1.3. Notice first that the event in (1.8) involves the supremun over time. In order to reduce to a single time estimate, so that we may apply the results of [4], we fix a time grid of length T_ε in the interval $[0, \varepsilon^{-2}T]$. T_ε is chosen so small such that the probability that more than two pairs of stirring particles at any site $|x| < 2\varepsilon^{-n}$ move in the same $(iT_\varepsilon, (i+1)T_\varepsilon]$ interval for some $i \in \{0, \ldots, [\varepsilon^{-2}T/T_\varepsilon]+1\}$ is vanishingly small in ε. T_ε should therefore satisfy the condition

$$\lim_{\varepsilon\to 0}(\varepsilon^{-n}T_\varepsilon)^2\varepsilon^{-2}T/T_\varepsilon = 0.$$

For instance we can take

$$T_\varepsilon = \varepsilon^{2n+3}. \tag{2.16}$$

It is then enough to prove that there is $\delta' > 0$ and for any $N \geqslant 1$ there is c_N so that for all $\varepsilon \in (0, 1)$,

$$\sup_{t\leqslant\varepsilon^{-2}T}\sup_{y\in\mathbb{Z}} E_{\mu^\varepsilon}\left[\left(\sum_{x\in I(\varepsilon,\beta;y)}\varepsilon^{\beta-\gamma}\prod_{i=1}^{k}\{\eta(x_i+x,t) - E_{\mu^\varepsilon}[\eta(x_i+x,t)]\}\right)^{2N}\right]$$

$$\leqslant c_N\varepsilon^{\delta' N}. \tag{2.17}$$

Equation (2.17) is easily obtained from Theorem 2.2 of [4] because $\gamma < \frac{1}{2}\beta$. □

Acknowledgments

M.E. Vares acknowledges very kind hospitality at the Mathematical Departments of the Universities of Rome.

References

[1] E. Andjel, A correlation inequality for the symmetric simple exclusion process, Ann. Probab. 16 (1988) 717-721.

[2] E. Andjel and C. Kipnis, Pointwise ergodic theorems for non ergodic interacting particle systems, Probab. Theory Rel. Fields 75 (1988) 545-550.

[3] A. DeMasi, N. Ianiro, A. Pellegrinotti and E. Presutti, A survey of the hydrodynamical behavior of many particle systems, in: J.L. Lebowitz and E.W. Montroll, eds., Nonequilibrium Phenomena II: From Stochastics to Hydrodynamics, Studies in Statistical Mechanics, Vol. XI, (North-Holland, Amsterdam, 1984) pp. 127-293.

[4] P.A. Ferrari, E. Presutti, E. Scacciatelli and M.E. Vares, The symmetric simple exclusion process, I: Probability estimates, Stochastic Process. Appl. 39 (1991) 89-105, this issue.

[5] T. Harris, Additive set valued Markov process and graphical methods, Ann Probab. 6 (1978) 355-378.

[6] T. Liggett, Interacting Particle Systems, (Springer, Berlin, 1985).